Lagrange Multipliers

Constrained optimization

Maximize $z = f(x,y)$
or $w = f(x,y,z)$
as before but now we have a constraint, on which (x,y) we can use.

Ex) Apple makes x ipads & y iphones, generating revenue of $f(x,y) = 8x+6y$.

The market is saturated when
$g(x,y) = x^2+y^2=4$.
How can we maximize revenue?

We want to find (x,y) satisfying $g(x,y)=4$, that maximizes $f(x,y)$.

Initial guess: $(x,y)=(2,0) \Rightarrow f(x,y)=16$

Can we do better?

Definitions:
- The objective function is $f(x,y)$
  (we want to maximize/minimize)
- The constraint is $g(x,y)=$ constant (fixed).

Ex in $\mathbb{R}^3$: what point on the plane

$3x+5y+7z=10$ is closest to the origin?

Objective: Minimize $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, or
we can make the algebra simpler by
minimizing $D(x,y,z) = x^2+y^2+z^2$ instead.

Constraint: $g(x,y,z) = 3x+5y+7z=10$.

Apple's goal: Maximize the objective function
$f(x,y) = 8x+6y$ subject to the constraint
$g(x,y) = x^2+y^2=4$.

Clever geometric idea
graph the constraint $g(x,y)=4$ in $\mathbb{R}^2$
(Note that this is a level curve of g)
Then graph the level curves of the objective function
$f = 8x+6y$.
We want the highest value of f we can get. That will occur when the level curves of f are just barely touching the constraint.
(so they are tangent hence parallel)

Remember that
∇f point $\perp$ to the curves of f.
∇f point $\perp$ to the curves of g.

So the level curves of f & g are parallel.

when their gradient vectors are parallel

So $\nabla f = \lambda \nabla g$ for some scalar λ "Landa"

(Aside: This technique is called
Lagrange multiplier, hence the λ.)

$\nabla f = \lambda \nabla g$ to find (x,y) (& λ).

Apple solution:

$$\nabla f = \langle 8,6 \rangle$$
$$\nabla g = \langle 2x, 2y \rangle$$

So we have $\langle 8,6\rangle = \lambda \langle 2x, 2y\rangle$

1. $8 = 2\lambda x$
2. $6 = 2\lambda y$
3. $x^2+y^2=4$

This gives us 2 equations in 3 unknowns (x, y, λ) but we have the 3rd equation.

Frequent technique for solving these systems:

Solve the 1st two equations for x & y in terms of λ. Then plug them into the constraint & solve for λ.

$x = \frac{4}{\lambda}$ & $y = \frac{3}{\lambda}$

$$\left(\frac{4}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 = 4 \implies 25 = 4\lambda^2$$

$$\frac{16}{\lambda^2} + \frac{9}{\lambda^2} = 4$$
$$\frac{25}{\lambda^2} = 4$$
$$25 = 4\lambda^2$$

$$\lambda = \pm \frac{5}{2}$$

$\lambda = \frac{5}{2} \implies x = \frac{8}{5}, \; y = \frac{3}{\frac{5}{2}} = \frac{6}{5}$

$\lambda = -\frac{5}{2} \implies x = -\frac{8}{5}, \; y = -\frac{6}{5}$

(we can't mix & match positive & negatives x's & y's)

So $(8/5, \frac{6}{5})$ gives the maximum &
$(-\frac{8}{5}, -\frac{6}{5})$ gives the minimum.

$f\left(\frac{8}{4}, \frac{6}{5}\right) = 20$ much better than the initial guess.

Common pitfall: How do you simplify something like $2x = \lambda y x$

If you divide by x

$\Rightarrow 2 = \lambda y$, you are throwing away possible solution coming from $x = 0$.

Safer way: set it to 0.

$$2x - \lambda y x = 0$$

$$x(2 - \lambda y) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad 2 - \lambda y = 0$$

Then pursue each possibility to different solutions.

<u>The orthogonal Gradient Theorem:</u>

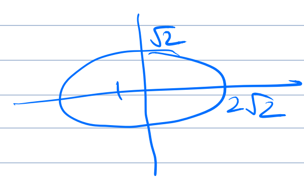
Suppose that $f(x,y,z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad r(t) = x(t)i + y(t)j + z(t)k$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

<u>Ex</u> Find the greatest & smallest values that the function

$$f(x,y) = xy$$



takes on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

Ans: We want to find the extreme values of $f(x,y) = xy$ subject to the constraint $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

To do so, we first find the values of x, y, λ for which

$$\nabla f = \lambda \nabla g \quad \& \quad g(x,y) = 0$$

The gradient equation

$$yi + xj = \frac{\lambda}{4} x i + \lambda y j$$

$\Rightarrow y = \frac{\lambda}{4} x$, $\quad x = \lambda y$ & $y = \frac{\lambda}{4}(\lambda y)$

$$= \frac{\lambda^2}{4} y$$

So that $y = 0$ or $\lambda = \pm 2$

Case 1: If $y = 0$, then $x = y = 0$

But $(0, 0)$ is not on the ellipse.

Case 2: If $y \neq 0$, $\Rightarrow$ $\lambda = \pm 2$ & $x = \pm 2y$

substitution this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1 \Rightarrow \frac{4y^2}{8} + \frac{y^2}{2} = 1$$

$$\frac{y^2}{4} + \frac{y^2}{2} = 1 \Rightarrow y^2 + 2y^2 = 4$$

$$3y^2 = 4$$

$$y^2 = \frac{4}{3}$$

$$y = \pm \frac{2}{\sqrt{3}}$$

The function $f(x, y) = xy$

$\therefore$ it takes on the extreme values on the ellipse at the four point.

$$\left(\pm \frac{4}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right), \left(\pm \frac{4}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$$

The extreme values are

$xy =$

$$\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right) = \frac{8}{9} \quad \& \quad \left(\frac{4}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{3}}\right) = \frac{-8}{9}$$

Lagrange Multiplier with 2 constraints

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ $\quad g_1(x,y,z)=0,$
$\quad g_2(x,y,z)=0$

3 variables

we can maximize $w = f(x,y,z)$ subject to $g(x,y,z) = \text{constant}$
but then you get 4 equations in 4 unknown.

$$\nabla f = \lambda \nabla g \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$$
$$f_z = \lambda g_z$$
$$g(x,y,z) = \text{constant.}$$

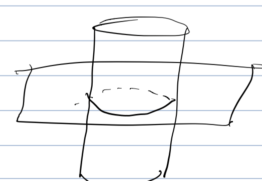
Classic example:
Make a rectangular fence along a river to enclose the largest possible field, using 200 meters of fence.

Solution: Maximize $f(x,y) = xy$
subject to constraint $g(x,y) = 2x + y = 200$
$\nabla f = \lambda \nabla g \qquad y i + x j = \lambda \langle 2,1 \rangle$

$2\lambda = y \qquad \lambda = x$

$2x + y = 200$
$4\lambda = 200 \qquad A = \boxed{5000\,m^2}$
$\lambda = 50$
then $x = 50$
$y = 100$

Ex) The plane $x+y+z=1$ cuts cylinder $x^2+y^2=1$ in an ellipse.
Find the points on the ellipse that lie closest to + farthest from the origin.



Ans: $f(x,y,z) = x^2+y^2=z^2$

(The square of the distance from (x,y,z) to the origin)
subject to the constraints.

$g_1(x,y,z) = x^2+y^2-1 = 0$
$g_2(x,y,z) = x+y+z-1=0$

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$\langle 2x, 2y, 2z\rangle = \lambda\langle 2x, 2y\rangle + \mu(i+j+k)$

$2xi + 2yj + 2zk = \lambda 2xi + \lambda 2yj + \mu i + \mu j + \mu k$

$2xi + 2yj + 2zk = \lambda 2xi + \mu i + \lambda 2yj + \mu j + \mu k$

$2x = \lambda 2x + \mu$, $2y = \lambda 2y + \mu$, $2z = \mu$

$2x = 2\lambda x + 2z \Rightarrow z = (1-\lambda)x$

$2y = 2\lambda y + 2z \Rightarrow z = (1-\lambda)y$

If either $\lambda = 1$ & $z = 0$ or $\lambda \neq 1$ & $x = y = \dfrac{z}{1-\lambda}$

If $\lambda = 1$ & $z = 0$, $\quad x^2+y^2-1=0 \quad x^2+y^2-1=0$

$x+y+z-1=0 \rightarrow x+y-1=0$ Solve for y

$\Rightarrow (1,0,0)$ & $(0,1,0)$ on the ellipse

If $\lambda \neq 1$ & $x = y$, Then

$x^2+y^2-1=0$

$x+y+z-1=0 \rightarrow 2x+z-1=0$

$z = 1-2x$

$x^2+y^2-1=0$

$x^2+x^2-1=0$

$2x^2 = 1$

$x = \pm\dfrac{\sqrt{2}}{2} \qquad z = 1 \mp \sqrt{2}$

$\rightarrow$ corresponding point.

$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2}\right)$

$P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$

Becareful, although P_1 & P_2 both give local maxima of f on the ellipse, P_2 is further from the origin than P_1.

The points on the ellipse closest to the origin are $(1,0,0)$ & $(0,1,0)$

& furthest $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$